Sixth Term Examination Paper [STEP]

Mathematics 2 [9470]

2018

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STEP MATHEMATICS 2

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Examiner's Report

General Comments

The pure questions were again the most popular of the paper, with only two of those questions being attempted by fewer than half of the candidates (none of the other questions was attempted by more than half of the candidates). Good responses were seen to all of the questions, but in many cases, explanations lacked sufficient detail to be awarded full marks. Candidates should ensure that they are demonstrating that the results that they are attempting to apply are valid in the cases being considered. In several of the questions, later parts involve finding solutions to situations that are similar to earlier parts of the question. In general candidates struggled to recognise these similarities and therefore spent a lot of time repeating work that had already been done, rather than simply observing what the result must be.

Question 1

This was the most popular question of the paper, attempted by 93% of candidates. While many candidates scored well on this question there were very few who achieved full marks and a significant number who scored 0 for this question.

The first two parts of the questions were generally done well, with the majority of marks being lost for arithmetic errors or from just considering one of the two cases. In some cases, candidates produced the reasoning for part (ii) as their answer for part (i). In these cases, the candidates generally repeated the work as their answer for part (ii). There were a number of solutions to part (ii) that were overcomplicated, involving consideration of the factorised form and comparison of coefficients.

Part (iii) was also very well completed by those who attempted it, with the main cause for loss of marks being algebraic errors. A number of errors were seen in the final part of the question, particularly not considering the second case or not realising that the quadratic would have a different discriminant in this case and a failure to check that the roots of the quadratic did not duplicate the repeated root already found in this case.

Question 2

This was the least popular of the pure questions on the paper, attempted by only 35% of the candidates. Very few candidates were able to score full marks and a fairly high proportion scored 0 on this question.

In the first part of the question, a significant number of candidates sketched convex functions and so had sketches that did not match the inequality. Most candidates were able to identify many of the key points required for the final section of this introductory part, but many could not explain it clearly enough for full marks in this section.

Most candidates made good attempts at the next section, but again the presentations often lacked enough clarity about how the inequalities were being linked together to be awarded full marks. In some cases, the choices for x_1 and x_2 were not within the interval (x_1, x_2) and so these candidates lost marks.

Candidates who attempted to complete the final parts of the question were generally confident about how the results from the first part were to be applied, but a significant number omitted to demonstrate that the function being used was concave and so were not able to achieve full marks here.

Question 3

The first part of this question was well attempted in general, although some solutions required more care to be taken to check whether or not the extreme values are included within the range or not.

Many solutions to part (ii) did not generally include very clear explanations of the method and candidates often did not make it clear that they had demonstrated the result in both directions.

For the final part, candidates often did not find the “certain point” referred to in the question and instead tried to work with a general point (a, b). Some candidates also attempted to calculate the integral directly rather than making use of the fact that the graph has rotational symmetry.

Question 4

This was the second most popular question on the paper. The average mark scored by candidates attempting this question was the highest of all the pure questions.

The first two parts of the question were generally well done with candidates showing confidence in applying the given identity and then factorising and solving the resulting equation. A sizeable minority struggled to deal with the given range for x and so were unable to find all the solutions. In part (ii) many candidates were able to explain clearly why $\cos x = \cos y$ leads to $x = y$.

Part (iii), however, proved to be difficult for the majority of candidates. A small number of candidates did manage to find the quadratic equation and most of these were able to proceed and complete the question fully. Most candidates did not score very many marks in this section.

Question 5

Of the pure questions this was the one that attracted the poorest responses in general, with a significant proportion of attempts scoring 0.

In the first part of the question the majority of candidates did not include the constant of integration and so did not produce a fully justified solution. The expansion and substitution in the second part of the question was done well in general, although many candidates attempted to expand the e^{-x} as well as the $(1 - e^{ax})$, after which they were unable to complete the integral. In many of these cases there were then unjustified jumps to the series expansion found in part (i).

Answers to part (iii) were generally much better than part (ii), although some substitution errors were seen. Most of the marks lost in this part were because candidates failed to spot the connection with the previous part. It is worth noting that many candidates did not attempt part (iii), having failed to complete part (ii) successfully. It is likely that some additional marks could have been scored by these candidates had they attempted this final part.

Question 6

Solutions on this question either scored very well or very poorly, depending on the quality of explanation provided by candidates in their solutions. In the weakest cases the only marks that were awarded were for finding some of the particular cases.

In the first part of the question candidates were generally able to find the cases that satisfied the equation, but many of the explanations that there are no solutions if $n \geq 5$ were not sufficiently well produced to receive full marks.

In the second part of the question many students just restated the theorems without explaining the reasoning that followed from them. The cases for small values of n were generally found, but some candidates struggled to find the pair $(4,10)$. Some candidates also did not attempt to explain why the cases $n = 5$ and $n = 6$ did not produce solutions.

There were some attempts to calculate the values of large factorials in this question. Candidates should be aware that such an approach will not be the correct method with which to tackle the questions.

Question 7

This was an unpopular question among the pure questions, with slightly less than two fifths of the candidates attempting it. Many of the responses did not progress far through the problem, although there were some excellent solutions seen. As a result, attempts at this question generally received either very few or most of the marks.

A range of different methods were used, such as consideration of ratios of areas, or parallel lines to the diagram and the use of similar triangles.

The most common difficulties arose from candidates not recognising that $\boldsymbol{m}$ was parallel to $\boldsymbol{a}$ and $\boldsymbol{n}$ was parallel to $\boldsymbol{b}$.

Question 8

This was the third most popular question on the paper and a number of very good responses were seen from candidates.

Most candidates were able to apply the given substitution to the differential equation and then separated the variables successfully. Candidates adopted a range of approaches to solving the differential equation, such as use of an integrating factor or a solution by finding a complementary function and then a particular integral. Success was seen with all of these methods. Some candidates omitted the constant of integration and then were unable to reach the correct answer.

In the second part of the question most candidates were able to spot a suitable substitution and proceeded to solve the differential equation successfully. The best candidates spotted the similarity to part (i) and therefore saved some time on this part by modifying the answer to (i) rather than working through all of the steps again.

The final part of the question was the least well answered. Although most candidates realised that $y_1(x) > y_2(x)$, many did not justify this or substituted a value to check rather than demonstrating that it was true for all values. Many candidates also failed to recognise that the gradient should be 0 at the origin for both curves.

Question 9

This was the most popular of the mechanics questions, with almost half of the candidates attempting it. Many candidates were able to apply the required sequence of calculations and secured good marks on this question, although in some cases some steps were omitted which limited the amount of success that could be achieved on the question. In the final part of the question weaker candidates struggled to get the signs of the velocities correct and therefore were unable to reach the correct coefficient of restitution between the two particles.

Question 10

The qualities of responses for this question were quite varied. Many candidates who struggled with this question failed to set up the speed of a point on the string in the first part of the question and were therefore unable to make any significant progress. In the part of the question considering the return journey a common error was to “restart the clock” once the ant had reached the endpoint, however, since the speed of the string is dependent on the time from the start of the problem this led to an error.

A particularly elegant solution to the final part involved changing the frame of reference such that the endpoint was stationary and the peg was moving at constant speed. Stronger candidates were then able to use the symmetry of the problem to reduce the amount of algebra needed considerably.

Question 11

The most common mistake made in this question was to have the frictional force acting in the wrong direction, with many candidates assuming that the frictional force was pulling the motorbike backwards and a “driving force” from the engine acted to push it forwards. The great majority of candidates did attempt to find moments about the centre of mass as instructed, but there were some attempts to evaluate moments about one of the wheels.

Question 12

This question was the most poorly attempted of all of the questions. While approximately one fifth of candidates attempted this question (more than questions 10 or 11), many of the solutions scored very low marks. The few candidates who were able to make progress on the question were however able to secure very good marks.

There were a number of candidates who clearly did not understand the payoffs described in this question, thinking for example that the process continued until a tail was reached and was then related to the number of heads achieved. Many of the students attempted differentiation to maximise the expected winnings, but often did not progress beyond the non-integer value found to check the two possible integer values.

In the second part many candidates struggled to find a useful method of counting successful outcomes, and therefore could not make much further progress on the question. In many cases the quality of explanation seen accompanying the method was not sufficiently detailed to demonstrate that a valid method was being attempted – candidates would be well advised to pay attention to the explanation of their method in questions such as this.

In the final part, many candidates failed to recognise the similarity between the function to be maximised and that from the first part of the question and therefore attempted to work through the process again. The manipulation of logarithms for the final part of the question was generally well done.

Question 13

This was the more popular of the two probability and statistics questions on the paper and many good responses were seen.

Candidates were generally able to work out the probabilities required in the first part of the question accurately. A considerable number of candidates were not able to make significant progress beyond that point, but those who did were often able to identify the relationships clearly and make use of the symmetry of the problem. Some attempts to tackle the second part of the question through counting arguments were seen, but these were not successful. The proportion of candidates achieving full marks for this question was higher than any other.

STEP MATHEMATICS 2

2018

Hints and Solutions

Question 1

The first result can be shown by substituting k^{-1} into the quartic expression.

It then follows for part (i) that the only way to achieve one distinct root is for that root to be either 1 or -1. In either case the factorised form of the quartic can then be considered to find the values of a and b.

Similarly, for three distinct roots, there must be one pair (k, k^{-1}) along with either 1 or -1. Substitution of 1 or -1 into the quartic then leads to the required relationships.

For part (iii) note that the case $b = 2a - 2$ corresponds to the case where there is a root of -1 and it can be seen that it must be a repeated root. The other factor is therefore a quadratic, which can then be solved.

Finally, the conditions for there to be three roots can be found by considering the discriminant of the quadratic (and the corresponding one for the other case). It is also necessary to confirm that this quadratic does not repeat the root of either 1 or -1 depending on which case is being considered.

Question 2

The point with x-coordinate $tx_1 + (1-t)x_2$ is a point within the range (x_1, x_2) (for $0 < t < 1$), and $tf(x_1) + (1-t)f(x_2)$ is the y-coordinate of a point on the chord joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Therefore, for any position between the two endpoints on the curve, the inequality is comparing the point on the curve with the point on the chord joining the two points. The sketch therefore needs to show the chord entirely below the curve. The final part of the introductory section can be shown either with a proof by contradiction or by arguing that $f''(x) < 0$ means that the gradient is always decreasing within the interval.

Part (i) requires choosing values for x_1 and x_2 so that the inequality can be applied and then applying this process multiple times to reach the required result. In each case the choice of x_1 and x_2 need to be made so that they lie within the range for which applying the inequality is valid.

Parts (ii) and (iii) both follow from the result of part (i), but it is important to check that the function being used is concave in the relevant range which needs to be stated clearly. In part (ii) the result follows immediately, whereas the final part requires some manipulation of logarithms to reach the final form of the relationship.

Question 3

The differentiation for the first part of the question can be achieved by applying the chain rule. Consideration of the sine function within the interval then allows the range to be determined. Consideration of the gradient function then allows the graph to be sketched - it can be seen from the symmetry of the sine function that $f'(x) = f'\left(\frac{1}{2}\pi\right) - x$, which means that the graph must have rotational symmetry about the point where $x = \frac{1}{4}\pi$.

A sketch of a rotationally symmetric function is a helpful way of demonstrating the second part as it allows the distances that must be equal to be identified clearly. It is important to show clearly that the result works in both the if and only if directions. For the final question in this part, note that the sections of the graph above the axis must exactly match those below the axis, so the area must be 0.

For part (iii), begin by showing that the equation from part (ii) holds for this function. Once the rotational symmetry has been demonstrated it follows that the area of any interval with the centre of rotation in the centre will be equal to the area of a rectangle over the same interval passing through the centre of rotation.

Question 4

For the first part, note that two of the terms in the left-hand side of the equation have a coefficient of 1 and two have a coefficient of 3. Applying the given identity to each of these pairs gives a common factor of $\cos\frac{5x}{2}$. The equation can therefore be factorised and then another application of the given identity will allow the full set of roots to be found.

The identity given at the start of the question can be applied to the first two terms of the left-hand side of the equation in part (ii) and the double angle formula can be applied to the $\cos 2x$. This then leads to an equation that can easily be factorised to show the required result. The range of possible values needs to be considered when considering the case where $\cos x = \cos y$.

For the final part a similar process to part (ii) can be used to create a quadratic function of $\cos\frac{1}{2}(x+y)$. Completing the square or considering a discriminant then allows the solutions to be found.

Question 5

For the first part of the question note that $\ln(1 + x)$ can be obtained by integrating $(1 + x)^{-1}$ and so the required expansion can be found by integrating the binomial expansion term by term. Note also, that the integration produces a constant, which needs to be shown to be 0.

In part (ii), the series expansion of e^{ax} can be obtained by adjusting the series expansion of e^x. To evaluate the integral, substitute the series expansion for the e^{ax}, but leave the e^{-x} unchanged. The integration can then be completed term by term.

For part (iii) note that a substitution of $u = -\ln x$ will transform the integral into one that can be expressed in terms of the integral in part (ii), which then allows the result to follow.

Question 6

For the first part, note that 5 will certainly be a factor of the left-hand side in any case where $n \geq 5$. It then remains to check the other cases one at a time.

For part (ii), first explain how the two theorems show that there will not be any solutions if $n \geq 7$, by showing that $m > 4n$ and so there must be a prime factor of the right-hand side that cannot exist in the product on the left-hand side.

The remaining cases then need to be checked one at a time, noting that the individual numbers within the product can be split into their prime factorisation and then combined differently to form the right-hand side.

Question 7

It is very useful to draw a diagram to represent the situation described at the start of this question. Defining the vectors $\boldsymbol{m}$ and $\boldsymbol{n}$ as scalar multiples of $\boldsymbol{a}$ and $\boldsymbol{b}$ (using two new unknowns) allows the position vector of Q to be written in two different ways. Since the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, the coefficients of these vectors can be equated and this then leads to the correct expression for $\boldsymbol{m}$.

A similar process then leads to an expression for the position vector of L in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, but since L lies on OB, the coefficient of $\boldsymbol{a}$ must be 0.

It then follows that $\lambda\mu < 1$ means that L lies on the segment OB.

Question 8

Making the substitution given reduces the differential equation into one for which it is easy to separate the variables. The two sides can then be integrated to find a general solution to the equation and then the boundary condition can be applied to find the required solution.

The differential equation in part (ii) is similar to the one from part (i), so a similar substitution should work (using a cube root rather than square root). The same process can then be followed as in part (i) to solve this differential equation.

In part (iii), the information that $\alpha = \beta$ can be used to simplify the equations being considered and then it can be seen that the two curves will both approach an asymptote at $y = 1$. We also know that the curves both pass through the origin and the differential equations show that the curves should both have gradient 0 at the origin. All that remains is to deduce the relative positions of the two curves by considering the behaviour of the exponential function.

Question 9

Since the two particles are released from rest the distance between them will remain constant until A reaches the ground. The height of B above the ground at this point can therefore be calculated.

Application of the uniform acceleration formulae will therefore give the speeds of A and B at the moment A hits the ground and the coefficient of restitution can then be used to calculate the speed with which A rebounds. Since they both continue to move under gravity the speed of B relative to A will remain constant and therefore the time until they collide can be calculated. Once the time is known one of the uniform acceleration formulae can then be used to determine the height at which the collision happens and the two speeds at this time.

For the final part another application of the uniform acceleration formulae can be used to find the velocity of A immediately after the collision. Conservation of momentum can then be used to find the velocity of B, although care needs to be taken at this stage with the signs of the terms. Finally, the velocities before and after the collision can be used to calculate the coefficient of restitution.

Question 10

Finding an expression for the length of the string at time t allows the speed of the point on the string to be determined. The differential equation can then be set up by adding the speed of the ant to the speed of the point on the string. The next result can then be verified by applying the quotient rule to perform the differentiation.

Once the differential equation has been verified, integration leads to a relationship between x and t, which then leads to the required result.

For the journey back, the differential equation needs to be changed so that the speed of the ant is subtracted rather than added. The differential equation can then be rewritten in a manner similar to the first part of the question and solved.

Question 11

A diagram representing the situation will help to ensure that the correct calculations are performed. In particular it is important to note that the frictional force will be acting in the direction of motion of the motorbike. Taking moments about the centre of mass as instructed and then setting the reaction at the front wheel of the motorbike to 0 for the case when the front wheel loses contact with the ground gives the maximum possible frictional force for this motion. Comparing this to μR then gives the first inequality.

When the rear wheel is about to slip the frictional force will be taking its maximum value. Substituting this into the equation found by taking moments and resolving forces vertically then allows the value of this frictional force to be found. Newton's second law then gives the acceleration.

For the final part, first show that the maximum acceleration is at the moment when the front wheel would be about to leave the ground. The value of the frictional force at this point can then be found and the acceleration can then be deduced.

Question 12

First note that the only winning sequence is h heads in a row and the probability of this can be found easily. The expected winnings can then be expressed as a function of h (E_h). By considering the value of $\frac{E_{h+1}}{E_h}$ it can be shown that the expected winnings increases until $h = N$, remain the same for the next case and then decreases thereafter.

For the second part there are multiple sequences that lead to a win. Begin with a sequence of h heads and then consider adding tails to any of the h positions before those heads (only 1 tail can be placed in each position). The number of ways of winning with a total of t tails in the sequence can therefore be seen to be $\binom{h}{t}$. The sum of these probabilities can then be seen to be a binomial expansion and can therefore be simplified. An expression for the expected winnings can therefore be found. The case where $N = 2$ leads to a function of the form of the previous part, so the point at which the maximum value occurs can be written down immediately. Taking logarithms of this maximum value allows it to be shown that the value is very close to 3^1.

Question 13

The probabilities in the first part of the question can most easily be deduced by using a tree diagram.

For the second part, note that the probabilities at B must be equal to the probabilities at D by the symmetry of the problem. The sum of the four probabilities for any value of n must be equal to 1. Therefore, it is possible to deduce a recurrence relation for B_n and see that this remains at a constant value. With the values of B_n and D_n known recurrence relations for A_n and C_n can be found. These recurrence relations can be related to geometric sequences in order to find the formula for the general term.

STEP MATHEMATICS 2

2018

Mark Scheme

Question 1

	Substitute $x = k^{-1}$ into the quartic expression:	**M1**
	$$k^{-4} + ak^{-3} + bk^{-2} + ck^{-1} + 1 = \frac{1 + ak + bk^2 + ak^3 + k^4}{k^4}$$	
	Since k cannot be 0 and the numerator is equal to 0 (since k is a root of the equation), k^{-1} must also be a solution to the equation.	**E1**
(i)	For there to be only one distinct root, the root must be either 1 or -1	
	If the root is 1 then $a = -4, b = 6$	**B1**
	If the root is -1 then $a = 4, b = 6$	**B1**
(ii)	For there to be three distinct roots there must be one repeated root (which must be either 1 or -1).	**E1**
	If the repeated root is $x = 1$ then:	**M1**
	$1 + a + b + a + 1 = 0$	
	Therefore $b = -2a - 2$	**A1 AG**
	If the repeated root is $x = -1$ then:	**M1**
	$1 - a + b - a + 1 = 0$	
	Therefore $b = 2a - 2$	**A1 AG**
(iii)	$b = 2a - 2$ corresponds to the case where the repeated root is -1.	
	$x^4 + ax^3 + bx^2 + ax + 1 = (x + 1)(x^3 + (a - 1)x^2 + (a - 1)x + 1)$	
	$(x + 1)$ is a factor of $(x^3 + (a - 1)x^2 + (a - 1)x + 1)$	
	$x^4 + ax^3 + bx^2 + ax + 1 = (x^2 + 2x + 1)(x^2 + kx + 1)$	**M1**
	Comparing coefficients of x^3:	**A1**
	$a = k + 2$	
	Therefore the other roots are	**M1**
	$\dfrac{(2 - a) \pm \sqrt{(a - 2)^2 - 4}}{2}$	**A1**
	In the case where $b = 2a - 2$:	
	For all three roots to be real, $(a - 2)^2 - 4 > 0$	**M1**
		A1
	$a^2 > 4a = 2b + 4$	
	In the case where $b = -2a - 2$, the quadratic will have $a = k - 2$	**M1**
	Therefore $(a + 2)^2 - 4 > 0$ for three roots	**A1**
	The quadratic factors in the two cases are both of the form $x^2 + kx + 1$. They	**M1**
	must have roots that are not ± 1.	**A1**
	$\frac{k \pm \sqrt{k^2 - 4}}{2} = \pm 1$ if $k^2 - 4 = (k \pm 2)^2$,	
	$(k \pm 2)^2 - (k + 2)(k - 2) = 0$, so $k = \pm 2$.	
	Therefore in neither of the two cases investigated does the quadratic equation have solutions of ± 1	
	Therefore	**A1**
	$(b + 2)^2 = 4a^2$	
	and	
	$a^2 > 2b + 4$	
	Are necessary and sufficient conditions for (*) to have exactly three distinct real roots.	

M1	Substituted correctly.
E1	Conclusion explained fully
B1	Values only need to be stated – there is no need to link them to the value of the root.
B1	Values only need to be stated – there is no need to link them to the value of the root.
	Subtotal: 4
E1	Identify that one of the roots must be 1 or -1
M1	Substitution of $x = 1$
A1	Conclude the first relationship
M1	Substitution of $x = -1$
A1	Conclude the second relationship
	Subtotal: 5
M1	Factorised form
A1	Comparison of coefficient
M1	Application of quadratic formula
A1	Correct roots
	Subtotal: 4
M1	Use of the discriminant
A1	$(a-2)^2 - 4 > 0$ and strictness explained
M1	Follow through same process for second case
A1	$(a+2)^2 - 4 > 0$
M1	Attempt to check that the roots of the quadratic are not equal to ± 1.
A1	Full justification.
A1	Any equivalent expression of the conditions
	Subtotal: 7

Question 2

	Sketch showing the curve and chord with the chord entirely below the curve and $f(x_1) < f(x_2)$	**E1**
	$tx_1 + (1-t)x_2$ identified as a value in the range (x_1, x_2)	**E1**
	$\big(tx_1 + (1-t)x_2, tf(x_1) + (1-t)f(x_2)\big)$ identified as the point on the chord.	**E1**
	If $f''(x) < 0$ for $a < x < b$ then the gradient of the curve $y = f(x)$ must be decreasing as x increases.	**E1**
	Suppose that a function $f(x)$ satisfies $f''(x) < 0$ for $a < x < b$, but is not concave for $a < x < b$. Then there must be points $x_1 < x_2$ and a value t, $0 < t < 1$ such that $tf(x_1) + (1-t)f(x_2) > f(tx_1 + (1-t)x_2)$	
	The gradient at $x = tx_1 + (1-t)x_2$ must be less than the gradient of the chord joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$, and so the curve $y = f(x)$ must continue to have a gradient of this value or less. The curve therefore cannot pass through $(x_2, f(x_2))$. Therefore, it must be the case that a function satisfying $f''(x) < 0$ for $a < x < b$ is concave for $a < x < b$.	**E1**
(i)	Let $x_1 = \frac{2u+v}{3}, x_2 = \frac{v+2w}{3}$ and $t = \frac{1}{2}$	**M1**
	Then, since $f(x)$ is concave for $a < x < b$: $\frac{1}{2}f\left(\frac{2u+v}{3}\right) + \frac{1}{2}f\left(\frac{v+2w}{3}\right) \le f\left(\frac{u+v+w}{3}\right)$	**A1**
	Setting $x_1 = u, x_2 = v$ and $t = \frac{2}{3}$ gives: $\frac{2}{3}f(u) + \frac{1}{3}f(v) \le f\left(\frac{2u+v}{3}\right)$	**B1**
	Similarly, setting $x_1 = v, x_2 = w$ and $t = \frac{1}{3}$ gives: $\frac{1}{3}f(v) + \frac{2}{3}f(w) \le f\left(\frac{v+2w}{3}\right)$	**B1**
	Therefore:	**M1**
	$f\left(\frac{u+v+w}{3}\right) \ge \frac{1}{2}f\left(\frac{2u+v}{3}\right) + \frac{1}{2}f\left(\frac{v+2w}{3}\right)$ $\ge \frac{1}{2}\left(\frac{2}{3}f(u) + \frac{1}{3}f(v)\right) + \frac{1}{2}\left(\frac{1}{3}f(v) + \frac{2}{3}f(w)\right) = \frac{f(u)+f(v)+f(w)}{3}$	**A1 AG**
(ii)	If $f(x) = \sin x$, then $f''(x) = -\sin x$ and $f''(x) < 0$ for $0 < x < \pi$.	**B1**
	Therefore $f(x)$ is concave for $0 < x < \pi$.	**E1**
	$0 < A, B, C < \pi$ and $A + B + C = \pi$, therefore, by (i): $\sin\frac{\pi}{3} \ge \frac{\sin A + \sin B + \sin C}{3}$	**M1**
	$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$	**A1 AG**
(iii)	If $f(x) = \ln(\sin x)$, then $f'(x) = \cot x$	**M1**
	$f''(x) = -\operatorname{cosec}^2 x$	**A1**
	Therefore $f''(x) < 0$ for $0 < x < \pi$ and so $f(x)$ is concave for $0 < x < \pi$	**E1**
	Therefore: $\ln\left(\sin\frac{\pi}{3}\right) \ge \frac{\ln(\sin A) + \ln(\sin B) + \ln(\sin C)}{3}$ $3\ln\left(\frac{\sqrt{3}}{2}\right) \ge \ln(\sin A \times \sin B \times \sin C)$	**M1**
	$\sin A \times \sin B \times \sin C \le \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$	**A1 AG**

E1	Sketch must match the case given in the question
E1	Could be explained in text or indicated on the graph (if clearly labelled)
E1	Could be explained in text
E1	Explanation includes reference to the behaviour of the gradient
E1	Fully clear explanation
	Subtotal: 5
M1	Any choice that will lead to $f\left(\frac{u+v+w}{3}\right)$ on RHS
A1	Application of definition of concave.
B1	Any choice that leads to an expression in terms of $f(u)$, $f(v)$ and $f(w)$ on LHS
B1	Any choice that leads to an expression in terms of $f(u)$, $f(v)$ and $f(w)$ on LHS
M1	Combination of previous inequalities
A1	Fully correct derivation
	Subtotal: 6
B1	States second derivative
E1	Concludes that the function is concave
M1	Application of result from (i) (including justification that it can be applied)
A1	Reaches correct inequality
	Subtotal: 4
M1	Differentiation of the correct function
A1	Correct second derivative
E1	Conclusion that the function is concave
M1	Application of result from (i)
A1	Correct manipulation of logarithms to reach given result.
	Subtotal: 5

Question 3

(i)	$f(x) = (1 + \tan x)^{-1}$	**M1**
	$f'(x) = -(1 + \tan x)^{-2} \sec^2 x$	**A1**
	$f'(x) = -\frac{1}{(1+\tan x)^2 \cos^2 x}$	
	$= -\frac{1}{(\sin x+\cos x)^2}$	**M1**
	$= -\frac{1}{\sin^2 x+\cos^2 x+2\sin x\cos x}$	**A1**
	$= -\frac{1}{1+\sin 2x}$	**AG**
	Within the given domain, $0 \leq \sin 2x \leq 1$, so $-1 \leq f'(x) \leq \frac{1}{2}$	**B1**
	Sketch of graph should have the following features:	
	Decreasing function	**G1**
	Points (0,1) and $\left(\frac{\pi}{2}, 0\right)$	**G1**
	Point of inflexion at $x = \frac{\pi}{4}$	**G1**
	All other features correct	**G1**
(ii)	If the point $(x, g(x))$ is rotated through 180 degrees about the point (a, b) then the image will be at the point $(a + (a - x), b + \left(b - g(x)\right))$.	**E1**
	Therefore, if the curve has rotational symmetry of order 2 about the point (a, b), then $g(2a - x) = 2b - g(x)$, so $g(x) + g(2a - x) = 2b$	**E1**
	Similarly, if $g(x) + g(2a - x) = 2b$, then any pair of points that are centred horizontally on the point (a, b) will also be centred vertically on the point (a, b),	**E1**
	which means that the curve will have rotational symmetry about that point.	**E1**
	$\int_{-1}^{1} g(x)\, dx = 0$	**B1**
(iii)	Since $\tan\left(\frac{\pi}{2} - x\right) = \cot x$,	**B1**
	$f\left(\frac{\pi}{2} - x\right) = \frac{1}{1+\cot^k x}$	**M1**
	$= \frac{\tan^k x}{\tan^k x+1} = 1 - f(x)$	**M1**
	Therefore $f(x) + f\left(2\left(\frac{\pi}{4}\right) - x\right) = 2\left(\frac{1}{2}\right)$	
	So the curve has rotational symmetry of order 2 about the point $(\frac{\pi}{4}, \frac{1}{2})$	**A1**
	The area under the curve over any interval centred on $x = \frac{\pi}{4}$, will therefore have the same area as a rectangle of the same width and height $\frac{1}{2}$.	**M1**
	Therefore $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1+\tan^k x}\, dx = \left(\frac{\pi}{3} - \frac{\pi}{6}\right) \times \frac{1}{2} = \frac{\pi}{12}$	**A1**

M1	Attempt to apply the chain or quotient rule
A1	Correct derivative
M1	Application of an appropriate trigonometric identity to simplify the function
A1	Fully correct simplification
B1	Correct range
	Subtotal: 5
G1	Feature clear on graph
G1	Feature clear on graph
G1	Feature clear on graph
G1	Feature clear on graph
	Subtotal: 4
E1	Identification of required image point
E1	Fully clear explanation
E1	Connection with points centred either horizontally or vertically on the correct value.
E1	Fully clear explanation
B1	Correct value
	Subtotal: 5
B1	Connection with cot, or application of an appropriate trigonometric identity
M1	Appropriate substitution to show rotational symmetry
M1	Correct manipulation to show rotational symmetry
A1	Rotational symmetry shown and point identified
M1	Equivalent area identified
A1	Correct value
	Subtotal: 6

Question 4

(i)	$\cos x + \cos 4x = 2\cos\frac{5}{2}x\cos\frac{3}{2}x$ and $\cos 2x + \cos 3x = 2\cos\frac{5}{2}x\cos\frac{1}{2}x$	**M1**
	$2\cos\frac{5}{2}x\cos\frac{3}{2}x + 6\cos\frac{5}{2}x\cos\frac{1}{2}x = 0$, so $2\cos\frac{5}{2}x\left(\cos\frac{3}{2}x + 3\cos\frac{1}{2}x\right) = 0$	**M1**
	Therefore $\cos\frac{5}{2}x = 0$ or $\cos\frac{3}{2}x + 3\cos\frac{1}{2}x = 0$	**A1**
	$\cos\frac{5}{2}x = 0$ gives $x = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}$ or $\frac{9\pi}{5}$	**B1 B1**
	If $\cos\frac{3}{2}x + 3\cos\frac{1}{2}x = 0$, then:	**M1**
	$\left(\cos\frac{3}{2}x + \cos\frac{1}{2}x\right) + 2\cos\frac{1}{2}x = 0$	
	$2\cos x\cos\frac{1}{2}x + 2\cos\frac{1}{2}x = 0$	
	$2\cos\frac{1}{2}x\,(\cos x + 1) = 0$	
	$\cos\frac{1}{2}x = 0$ or $\cos x = -1$, both of which give no new solutions to the equation.	**A1**
(ii)	$\cos(x-y) + \cos(x+y) = 2\cos x\cos y$	**M1**
	$2\cos x\cos y - 2\cos^2 x + 1 = 1$	**M1**
	$2\cos x\,(\cos y - \cos x) = 0$	**M1**
	Therefore either $\cos x = \cos y$, which can only be the case if $x = y$ since $0 \le x \le \pi$ and $0 \le y \le \pi$	**E1**
	Or $\cos x = 0$, so $x = \frac{\pi}{2}$	**A1**
(iii)	$2\cos\dfrac{1}{2}(x+y)\cos\dfrac{1}{2}(x-y)$	**M1**
	$-\left(2\cos^2\dfrac{1}{2}(x+y) - 1\right) = \dfrac{3}{2}$	**M1**
	$4\cos^2\dfrac{1}{2}(x+y) - 4\cos\dfrac{1}{2}(x+y)\cos\dfrac{1}{2}(x-y) + 1 = 0$	**M1**
	$\left(2\cos\dfrac{1}{2}(x+y) - \cos\dfrac{1}{2}(x-y)\right)^2 + 1 - \cos^2\dfrac{1}{2}(x-y) = 0$	**M1**
	$\left(2\cos\dfrac{1}{2}(x+y) - \cos\dfrac{1}{2}(x-y)\right)^2 + \sin^2\dfrac{1}{2}(x-y) = 0$	**M1**
	Therefore, since both terms are ≥ 0, they must both be equal to 0.	**M1**
	For $0 \le x \le \pi$ and $0 \le y \le \pi$, $\sin^2\frac{1}{2}(x-y) = 0$ only when $x = y$	**M1**
	Therefore $2\cos x = 1$, so $x = \frac{\pi}{3}$ and $y = \frac{\pi}{3}$	**A1**

M1	Pairing of terms in the equation
M1	Factorisation
A1	Identification of the two cases
B1	One solution identified
B1	Full set of solutions for first case
M1	Manipulation of equation from other case
A1	Justification that this gives no other roots to the equation
	Subtotal: 7
M1	Simplification of sum of cos functions or use of a compound angle formula
M1	Use of $\cos 2x$ identity
M1	Factorisation
E1	Explanation that $x = y$ (must refer to range of values for x and y)
A1	Correct value
	Subtotal: 5
M1	Simplication of sum of first two functions
M1	Use of $\cos 2A$ identity
M1	Simplification to three-term quadratic
M1	Completion of square, or calculation of discriminant
M1	Expression using sin function
M1	Explanation that this implies both equal
M1	Conclusion that $x = y$
A1	Correct solution
	Subtotal: 8

Question 5

(i) n^{th} term of expansion is $\frac{(-1)(-2)...(-n)}{n!}(x)^n$

$$(1+x)^{-1} = \sum_{n=0}^{\infty}(-x)^n$$ **B1**

$$\int (1+x)^{-1}\,dx = \ln(1+x) + c$$ **B1**

E1

$$\int \sum_{n=0}^{\infty}(-x)^n\,dx = -\sum_{n=0}^{\infty}\frac{(-x)^{n+1}}{n+1} = -\sum_{n=1}^{\infty}\frac{(-x)^n}{n}$$ **B1**

AG

(ii)
$$e^{-ax} = \sum_{n=0}^{\infty}\frac{(-ax)^n}{n!}$$ **B1**

$$\frac{(1-e^{-ax})e^{-x}}{x} = -\left(\sum_{n=1}^{\infty}\frac{(-a)^n}{n!}x^{n-1}\right)e^{-x}$$ **M1**

Let

$$I_n = \int_0^{\infty} x^n e^{-x}\,dx$$

$u = x^n \qquad \frac{dv}{dx} = e^{-x}$ **M1**

A1

$\frac{du}{dx} = nx^{n-1} \qquad v = -e^{-x}$

$$I_n = [-x^n e^{-x}]_0^{\infty} + n\int_0^{\infty} x^{n-1}e^{-x}\,dx$$ **M1**

$I_n = nI_{n-1}$ **M1**

$$I_0 = \int_0^{\infty} e^{-x}\,dx = [-e^{-x}]_0^{\infty} = 1$$ **M1**

Therefore $I_n = n!$ **A1**

So **A1**

$$\int_0^{\infty} -\left(\sum_{n=1}^{\infty}\frac{(-a)^n}{n!}x^{n-1}\right)e^{-x}\,dx = -\left(\sum_{n=1}^{\infty}\frac{(-a)^n}{n}\right) = \ln(1+a)$$ **AG**

(by part (i))

(iii) Let $u = -\ln x$ **M1**

Then $x = e^{-u}$ and $\frac{dx}{du} = -e^{-u}$ **M1**

Change limits:

$x = 1$ becomes $u = 0$

$x = 0$ becomes $u = \infty$ **B1**

$$\int_0^1 \frac{x^p - x^q}{\ln x}\,dx = \int_{\infty}^0 \frac{(e^{-pu} - e^{-qu})}{-u}(-e^{-u})du$$ **M1**

A1

$$= -\int_0^{\infty}\frac{(1-e^{-qu}) - (1-e^{-pu})}{u}(e^{-u})du$$ **M1**

$= -\ln(1+q) + \ln(1+p)$

$$= \ln\left(\frac{1+p}{1+q}\right)$$ **A1**

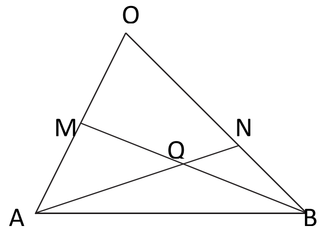
B1	Simplified form
B1	Correct integral
E1	Show that $c = 0$
B1	Correct integration term by term (ensure that signs are dealt with correctly)
	Subtotal: 4
B1	Correct expansion
M1	Substitution into the function to be integrated
M1	Integration by parts
A1	Correct derivative and integral
M1	Completion of integration by parts
M1	Simplification, including substitution of limits
M1	First case evaluated
A1	General result
A1	Fully correct solution
	Subtotal: 9
M1	Selection of appropriate substitution
M1	Differentiation
B1	Limits changed
M1	Substitution applied to the integral
A1	Completed substitution
M1	Rearrangement so that previous result can be applied
A1	Application of previous result (final simplification not needed)
	Subtotal: 7

Question 6

(i)	If $n \geq 5$ then $n! + 5 > 5$ and has 5 as a factor	**E1**
	Therefore the only possible solutions will have $n < 5$	**E1**
	The only pairs are therefore	
	$(2,7)$	**B1**
	$(3,11)$	**B1**
	$(4,29)$	**B1**
(ii)	If $n \geq 7$ then theorem 1 shows that $m > 4n$.	**E1**
	By theorem 2, there is a prime number between $2n$ and m, which must be a factor of $m!$	**E1**
	But that prime cannot be a factor of any of $1!, 3!, \cdots, (2n-1)!$	**E2**
	So it cannot be a factor of $1! \times 3! \times \cdots \times (2n-1)!$	**E1**
	Therefore there is a prime factor on the RHS that does not appear on the LHS.	**E1**
	Therefore the only pairs must have $n < 7$	**E1**
	$n = 1: m = 1$	**B1**
	$n = 2: m = 3$	**B1**
	$n = 3$: LHS=$3! \times 5!$	
	$3! \times 5! = 5! \times 6 = 6!$	
	So $m = 6$	**B1**
	$n = 4$: LHS= $3! \times 5! \times 7!$	**M1**
	$3! \times 5! = 2 \times 3 \times 2 \times 3 \times 4 \times 5 = (2 \times 4) \times (3 \times 3) \times (2 \times 5)$	
	So $m = 10$	**A1**
	$n = 5$: LHS = $3! \times 5! \times 7! \times 9!$	**E1**
	There must be two factors of 7 in the RHS, so $m \geq 14$	
	There will be no way of generating a factor of 11 for the RHS.	
	$n = 6$: LHS = $3! \times 5! \times 7! \times 9! \times 11!$	**E1**
	There must be two factors of 7 in the RHS, so $m \geq 14$	
	There will be no way of generating a factor of 13 for the RHS	**E1**

E1	Identification of common factor of 5
E1	No solutions for high values of n justified
B1	Correct solution
B1	Correct solution
B1	Correct solution
	Subtotal: 5
E1	Significance of theorem 1 explained
E1	Significance of theorem 2 explained
E2	Explicit statement that the prime cannot be a factor is required
E1	Can imply previous mark
E1	Prime factor on one side but not the other clearly explained
E1	Justification that solutions only exist for $n < 7$
	Subtotal: 7
B1	Correct solution
B1	Correct solution
B1	Correct solution
M1	Rearrangement of middle values to create $8 \times 9 \times 10$
A1	Correct value
E1	Explanation that $m \geq 11$
E1	Explanation that $m \geq 13$
E1	Identification that no factor of 13 exists in the LHS – can also be awarded for identifying that no factor of 11 exists in the LHS for previous case
	Subtotal: 8

Question 7



B1

Let k and l be such that $\boldsymbol{m} = k\boldsymbol{a}$ and $\boldsymbol{n} = l\boldsymbol{b}$

$\overrightarrow{BM} = k\boldsymbol{a} - \boldsymbol{b}$ **M1**

$\overrightarrow{QM} = \dfrac{\mu}{1+\mu}(k\boldsymbol{a} - \boldsymbol{b})$ **A1**

Similarly: **A1**

$\overrightarrow{QN} = \dfrac{\nu}{1+\nu}(l\boldsymbol{b} - \boldsymbol{a})$

Therefore: **M1**

$\boldsymbol{q} = \overrightarrow{OM} + \overrightarrow{MQ} = k\boldsymbol{a} - \dfrac{\mu}{1+\mu}(k\boldsymbol{a} - \boldsymbol{b})$

$\boldsymbol{q} = \dfrac{k}{1+\mu}\boldsymbol{a} + \dfrac{\mu}{(1+\mu)}\boldsymbol{b}$ **A1**

And: **M1**

$\boldsymbol{q} = \dfrac{l}{1+\nu}\boldsymbol{b} + \dfrac{\nu}{(1+\nu)}\boldsymbol{a}$ **A1**

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel: **M1**

$\dfrac{k}{1+\mu} = \dfrac{\nu}{(1+\nu)}$

Therefore **A2 AG**

$k = \dfrac{(1+\mu)\nu}{1+\nu}$

So

$\boldsymbol{m} = \dfrac{(1+\mu)\nu}{1+\nu}\boldsymbol{a}$

$\overrightarrow{AN} = \dfrac{(1+\nu)\mu}{1+\mu}\boldsymbol{b} - \boldsymbol{a}$ **B1**

Therefore: **M1**

$\overrightarrow{OL} = \dfrac{(1+\mu)\nu}{1+\nu}\boldsymbol{a} + p\left(\dfrac{(1+\nu)\mu}{1+\mu}\boldsymbol{b} - \boldsymbol{a}\right)$ **A1**

For some value of p

Since $\overrightarrow{OL}$ is parallel to $\boldsymbol{b}$, the coefficient of $\boldsymbol{a}$ must be 0 **M1**

$\dfrac{(1+\mu)\nu}{1+\nu} - p = 0$ **M1**

Therefore **M1**

$\overrightarrow{OL} = \dfrac{(1+\mu)\nu}{1+\nu}\boldsymbol{a} + \dfrac{(1+\mu)\nu}{1+\nu}\left(\dfrac{(1+\nu)\mu}{1+\mu}\boldsymbol{b} - \boldsymbol{a}\right) = \nu\mu\boldsymbol{b}$

So $\lambda = \mu\nu$ **A2**

$\mu\nu < 1$ means that L lies on OB. **E1**

B1	Diagram
M1	Method to work out $\overrightarrow{BM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$
A1	Expression for $\overrightarrow{QM}$
A1	Expression for $\overrightarrow{QN}$
M1	Find an expression for $\boldsymbol{q}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$
A1	Correct expression
M1	Find a second expression for $\boldsymbol{q}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$
A1	Correct expression
M1	Equate coefficients of $\boldsymbol{a}$
A2	Reach given expression for $\boldsymbol{m}$
	Subtotal: 11
B1	Find expression for $\overrightarrow{AN}$
M1	Form an equation of the line on which L lies.
A1	Correct equation
M1	Identify that the component in the direction of $\boldsymbol{a}$ must be 0
M1	Correct equation for p
M1	Substitution back into equation of line
A2	Correct relationship
E1	Correct explanation
	Subtotal: 9

Question 8

(i)

$$\frac{dv}{dt} = \frac{1}{2}y^{-\frac{1}{2}} \times \frac{dy}{dt}$$ **M1**

$$\frac{dy}{dt} = 2v\frac{dv}{dt}$$ **A1**

$$2v\frac{dv}{dt} = \alpha v - \beta v^2$$

$$\frac{dv}{dt} = \frac{1}{2}(\alpha - \beta v)$$ **M1**

$$\int \frac{1}{\alpha - \beta v}dv = \int \frac{1}{2}dt$$ **M1**

$$-\frac{1}{\beta}\ln|\alpha - \beta v| = \frac{1}{2}t + c$$ **M1**

$$\alpha - \beta v = Ae^{-\frac{1}{2}\beta t}$$ **M1**

$$v = \frac{1}{\beta}\left(\alpha - Ae^{-\frac{1}{2}\beta t}\right)$$ **A1**

$$y = \frac{1}{\beta^2}\left(\alpha - Ae^{-\frac{1}{2}\beta t}\right)^2$$

$$y_1 = \frac{\alpha^2}{\beta^2}\left(1 - e^{-\frac{1}{2}\beta t}\right)^2$$ **A1**

(ii)

Use the substitution $v = y^{\frac{1}{3}}$: **M1**

$$\frac{dv}{dt} = \frac{1}{3}y^{-\frac{2}{3}} \times \frac{dy}{dt}$$

$$3v^2\frac{dv}{dt} = \alpha v^2 - \beta v^3$$

$$\frac{dv}{dt} = \frac{1}{3}\alpha - \frac{1}{3}\beta v$$ **A1**

$$\int \frac{1}{\alpha - \beta v}dv = \int \frac{1}{3}dt$$

$$-\frac{1}{\beta}\ln|\alpha - \beta v| = \frac{1}{3}t + c$$

$$\alpha - \beta v = Ae^{-\frac{1}{3}\beta t}$$

$$v = \frac{1}{\beta}\left(\alpha - Ae^{-\frac{1}{3}\beta t}\right)$$ **A1**

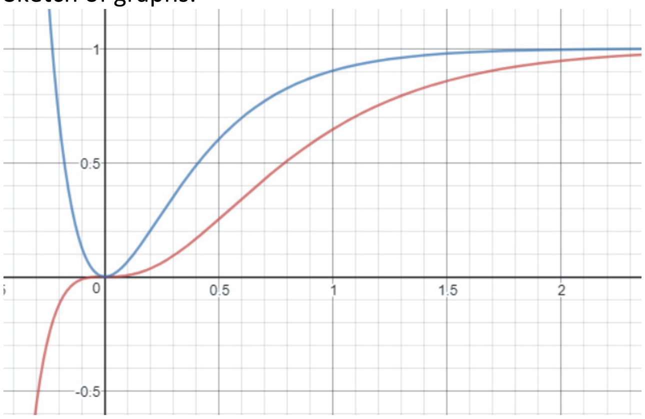
$$y = \frac{1}{\beta^3}\left(\alpha - Ae^{-\frac{1}{3}\beta t}\right)^3$$

$$y_2 = \frac{\alpha^3}{\beta^3}\left(1 - e^{-\frac{1}{3}\beta t}\right)^3$$ **A1**

(iii) If $\alpha = \beta$: **B1**

$y_1(x) = \left(1 - e^{-\frac{1}{2}\beta x}\right)^2$ and $y_2(x) = \left(1 - e^{-\frac{1}{3}\beta x}\right)^3$

Sketch of graphs:



Ignore anything to left of y-axis.

Both curves have a horizontal asymptote $y = 1$ **G1**

Both curves have gradient 0 as they pass through the origin **G1**

Both functions have decreasing gradient. **G1**

For positive values of x:

$0 > e^{-\frac{1}{3}\beta x} > e^{-\frac{1}{2}\beta x}$ **E1**

Therefore

$\left(1 - e^{-\frac{1}{3}\beta x}\right) < \left(1 - e^{-\frac{1}{2}\beta x}\right) < 1$ **E1**

$\left(1 - e^{-\frac{1}{3}\beta x}\right)^3 < \left(1 - e^{-\frac{1}{3}\beta x}\right)^2 < \left(1 - e^{-\frac{1}{2}\beta x}\right)^2$ **E1**

So the graph of y_2 should lie below the graph of y_1 **G1**

M1	Relationship between $\frac{dy}{dt}$ and $\frac{dv}{dt}$ (accept $dy = 2vdv$)
A1	Correct differentiation
M1	Substitution completed and simplified
M1	Variables separated
M1	Integration completed
M1	Logarithm removed
A1	Rearranged so that v is subject
A1	Formula for y and boundary condition applied (must be in the form $y = \cdots$)
	Subtotal: 8
M1	Correct substitution chosen and applied (could use $v = y^{\frac{2}{3}}$)
A1	Simplified differential equation reached
A1	Solution rearranged so that v is the subject
A1	Formula for y and boundary condition applied (must be in the form $y = \cdots$)
	Subtotal: 4
B1	Simplified expressions found for the case $\alpha = \beta$
G1	Asymptote must be indicated (accept if not explicit, but $y \rightarrow 1$ seen and clear from shape)
G1	Zero gradient through origin must be clear
G1	General shape away from origin correct. Accept any increasing function with decreasing gradient
E1	Comparison of exponential functions
E1	Comparison of the functions that will be raised to a power
E1	Correctly deduced relationship between the two graphs
G1	$y_1 > y_2$. Must have at least one of the E marks awarded to receive this mark.
	Subtotal: 8

Question 9

When A reaches the ground for the first time B will be at a height of $9h$ above P. **B1**
For the motion until A reaches the ground: **M1**
$u = 0, a = g, s = 8h$
$v^2 = u^2 + 2as$
$v^2 = 16gh$
Therefore $v = 4\sqrt{gh}$ **A1**
A rebounds with a speed of $2\sqrt{gh}\ ms^{-1}$ **A1**
The velocity of B relative to A for the subsequent motion will be $6\sqrt{gh}$ **B1**
The particles will therefore collide after $\frac{9h}{6\sqrt{gh}} = \frac{3h}{2\sqrt{gh}}\ s$ **M1** **A1**

For particle A:

$$u = -2\sqrt{gh}, a = g, t = \frac{3h}{2\sqrt{gh}}$$

$$s = ut + \frac{1}{2}at^2 = -2\sqrt{gh}\left(\frac{3h}{2\sqrt{gh}}\right) + \frac{1}{2}g\left(\frac{3h}{2\sqrt{gh}}\right)^2$$ **M1**

$$s = -3h + \frac{9h}{8} = -\frac{15}{8}h$$ **A1** **AG**

So the collision occurs a distance of $\frac{15}{8}h$ above P.

$$v = u + at = -2\sqrt{gh} + g\left(\frac{3h}{2\sqrt{gh}}\right)$$ **M1**

$$v = -\frac{1}{2}\sqrt{gh}$$ **A1**

$$u_A = \frac{1}{2}\sqrt{gh}$$

The velocity of B will be **M1**

$$-\frac{1}{2}\sqrt{gh} + 6\sqrt{gh} = \frac{11}{2}\sqrt{gh}$$

$$u_B = \frac{11}{2}\sqrt{gh}$$ **A1**

To hit the ground the second time with speed $4\sqrt{gh}$:

$v = 4\sqrt{gh}, a = g, s = \frac{15}{8}h$

$v^2 = u^2 + 2as$

$16gh = u^2 + \frac{15}{4}gh$ **M1**

$u^2 = \frac{49}{4}gh$ **A1**

$u = \frac{7}{2}\sqrt{gh}$ (since $u > -\frac{1}{2}\sqrt{gh}$) **E1**

Conservation of momentum for collision between the beads: **M1**

$m\left(-\frac{1}{2}\sqrt{gh}\right) + m\left(\frac{11}{2}\sqrt{gh}\right) = m\left(\frac{7}{2}\sqrt{gh}\right) + mv$

where v is the velocity of B after the collision.

$v = \frac{3}{2}\sqrt{gh}$ **A1**

$e = \frac{\frac{7}{2}\sqrt{gh} - \frac{3}{2}\sqrt{gh}}{\frac{11}{2}\sqrt{gh} - \left(-\frac{1}{2}\sqrt{gh}\right)} = \frac{1}{3}$ **M1** **A1**

B1	May be implied by later work
M1	Application of correct formula
A1	Correct value for velocity
A1	Correct rebound speed
B1	May be implied by later work
M1	Application of correct formula
A1	Correct time
M1	Application of correct formula
A1	Correct solution
	Subtotal: 9
M1	Application of correct formula
A1	Correct speed
M1	Application of correct formula
A1	Correct speed
	Subtotal: 4
M1	Application of correct formula
A1	Reach two possible values
E1	Select correct value
M1	Apply conservation of momentum
A1	Find velocity of B after collision
M1	Apply correct formula
A1	Correct value
	Subtotal: 7

Question 10

At time t the string will have a length of $a + ut$ **M1**

The speed of the point on the string will therefore be $\frac{xu}{a+ut}$ **A1**

$$\frac{dx}{dt} = \frac{xu}{a+ut} + v$$ **B1**

$$\frac{d}{dt}\left(\frac{x}{a+ut}\right) = \frac{(a+ut)\frac{dx}{dt} - xu}{(a+ut)^2}$$ **M1** **A1**

$$= \frac{xu + v(a+ut) - xu}{(a+ut)^2} = \frac{v}{a+ut}$$ **M1** **A1 AG**

$$\frac{x}{a+ut} = \int \frac{v}{a+ut}\,dt$$ **M1**

$$\frac{x}{a+ut} = \frac{v}{u}\ln|C(a+ut)|$$ **A1**

At $t = 0$, $x = 0$:

$$0 = \frac{v}{u}\ln aC$$ **M1**

Therefore $\mathrm{C} = \frac{1}{a}$ **A1**

At $t = T$, $x = a + uT$:

$$\frac{a+uT}{a+uT} = \frac{v}{u}\ln\left|\frac{1}{a}(a+uT)\right|$$ **M1**

$$1 + \frac{uT}{a} = e^k$$

where $k = u/v$.

$$uT = a(e^k - 1)$$ **A1 AG**

For the journey back:

$$\frac{dx}{dt} = \frac{xu}{a+ut} - v$$ **M1**

$$\frac{d}{dt}\left(\frac{x}{a+ut}\right) = -\frac{v}{a+ut}$$ **M1**

Therefore

$$\frac{x}{a+ut} = -\frac{v}{u}\ln|C(a+ut)|$$ **A1**

At $t = T$, $x = a + uT$:

$$\frac{a+uT}{a+uT} = -\frac{v}{u}\ln|C(a+uT)|$$ **M1**

Therefore:

$$C(a+uT) = e^{-k}$$ **A1**

Solve for $x = 0$:

$$0 = -\frac{v}{u}\ln|C(a+ut)|$$ **M1**

Therefore

$$C(a+ut) = 1$$

$$e^{-k}(a+ut) = a + uT$$

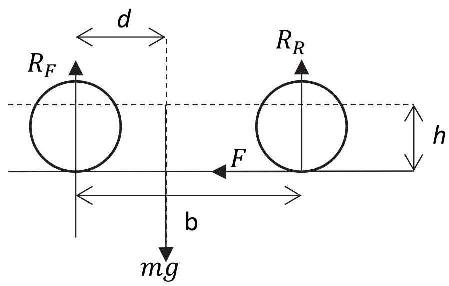
$$t = \frac{(a+uT)e^k - a}{u}$$

Therefore the time for the journey back is:

$$\frac{(a+uT)e^k - a}{u} - \frac{a\left(e^k - 1\right)}{u} = Te^k$$ **A1** **CAO**

M1	Expression for length at time t
A1	Correct speed
B1	Correct differential equation
M1	Use of quotient rule
A1	Correctly completed
M1	Substitution
A1	Result verified
	Subtotal: 7
M1	Method for solving the differential equation
A1	Correctly integrated
M1	Substitute boundary condition
A1	Correct value for constant
M1	Substitute for end point
A1	Simplified
	Subtotal: 6
M1	New differential equation
M1	Correct new differential
A1	Correct solution to differential equation
M1	Substitute for start of journey back
A1	Correct constant
M1	Solve for time of return
A1	Find time for return journey.
	Subtotal: 7

Question 11



Taking moments about the centre of mass: **M1**

$R_F d + Fh = R_R(b-d)$ **A1**

A1

$$F = \frac{R_R(b-d) - R_F d}{h}$$ **M1** **A1**

At the time when the front wheel loses contact with the ground: **B1**

$R_F = 0$ and $R_R = mg$

$$F = \frac{mg(b-d)}{h}$$

Maximum possible frictional force is μmg **B1**

Therefore if **E1**

$$\mu mg < \frac{mg(b-d)}{h}$$ **AG**

then the rear wheel will have slipped before this point.

i.e. if

$$\mu < \frac{b-d}{h}$$

At the moment before the rear wheel slips, friction will take its maximum value **B1**

$$\frac{R_R(b-d) - R_F d}{h} = \mu R_R$$ **M1**

Resolving vertically: **M1**

$R_F + R_R = mg$

$R_R b - mgd = \mu h R_R$ **M1**

$$R_R = \frac{mgd}{b - \mu h}$$ **A1**

Therefore

$$F = \frac{\mu mgd}{b - \mu h}$$

Newton's second law: **M1**

$F = ma$

Therefore **A1**

$$a = \frac{\mu dg}{b - \mu h}$$

The front wheel would lose contact with the road when $R_F = 0$:

The acceleration is given by **E1**

$$a = \frac{R_R b - mgd}{mh}$$ **E1**

Therefore a increases as R_R increases and R_F decreases

So the maximum acceleration is at the moment when the front wheel would be about to leave the ground **E1**

At this point $F = \frac{mg(b-d)}{h}$ and so **A1**

$$a = \frac{g(b-d)}{h}$$

B1	Forces all identified
M1	Taking moments
A1	All clockwise moments correct
A1	All anticlockwise moments correct
M1	Rearrange to make F the subject
A1	Correct form
B1	Identify reaction forces for this case
B1	Identify maximum possible value for F
E1	Explanation that rear wheel would have slipped
	Subtotal: 9
B1	Maximum value used
M1	Substituted into equation
M1	Resolve forces vertically (may be seen earlier)
M1	Eliminate R_F
A1	Correct reaction force
M1	Substitute into frictional force and apply Newton's second law
A1	Correct value for a
	Subtotal: 7
E1	Use of formula for the acceleration
E1	Identify that higher accelerations have higher reaction at the rear
E1	Identify moment when maximum acceleration occurs
A1	Correct value
	Subtotal: 4

Question 12

(i) I will win if there are h consecutive heads and lose otherwise. **M1**

$P(h\ consecutive\ heads) = p^h \left[= \left(\frac{N}{N+1}\right)^h\right]$

$Expected\ winnings = p^h h \left[= \left(\frac{N}{N+1}\right)^h h\right]$ **A1**

Let E_h be the expected winnings when the value h is chosen. **M1**

$$\frac{E_{h+1}}{E_h} = \left(\frac{N}{N+1}\right)\left(\frac{h+1}{h}\right) = \frac{Nh+N}{Nh+h}$$ **A1**

Therefore $\frac{E_{h+1}}{E_h} > 1$ if $h < N$ **M1**

And $\frac{E_{h+1}}{E_h} < 1$ if $h > N$ **M1**

So as h increases, the values of E_h increase until $h = N$, the value then remains the same for $h = N + 1$ and decreases thereafter. **A1**

So I can maximise my winnings by choosing $h = N$

(ii) Possible sequences that lead to a win are:

All heads: Probability: $\left(\frac{N}{N+1}\right)^h$

There are h positions available (one before each of the heads) where at most one tail can be placed. **B1**

1 tail can be placed in any of the h positions, so the probability of a sequence containing just one tail is $\binom{h}{1}\left(\frac{N}{N+1}\right)^h\left(\frac{1}{N+1}\right)^1$ **M1**

Similarly, for any other number of tails, $t \leq h$, the probability of a winning sequence containing that number of tails will be $\binom{h}{t}\left(\frac{N}{N+1}\right)^h\left(\frac{1}{N+1}\right)^t$ **M1**

Therefore the probability that I win is **M1**

$$\sum_{t=0}^{h}\binom{h}{t}\left(\frac{N}{N+1}\right)^h\left(\frac{1}{N+1}\right)^t = \left(\frac{N}{N+1}\right)^h\sum_{t=0}^{h}\binom{h}{t}\left(\frac{1}{N+1}\right)^t$$ **A1**

As the sum in the expression on the right is a binomial expansion it can be rewritten as $\left(\frac{1}{N+1}+1\right)^h$ **M1** **A1**

The probability that I win is therefore

$$\left(\frac{N}{N+1}\right)\left(\frac{1}{N+1}+1\right)^h = \frac{N^h(1+N+1)^h}{(N+1)^{2h}} = \frac{N^h(N+2)^h}{(N+1)^{2h}}$$

So my expected winnings are $\frac{hN^h(N+2)^h}{(N+1)^{2h}}$ **A1 AG**

In the case $N = 2$, the expected winnings are $h\left(\frac{8}{9}\right)^h$

The maximum value is when $h = 8$ or $h = 9$ and has a value of $\frac{8^9}{9^8}$ **B1**

$\log_3\left(\frac{8^9}{9^8}\right) = 9\log_3 8 - 8\log_3 9$ **M1**

$= 27\log_3 2 - 16$ **M1**

$\approx 27(0.63) - 16 = 1.01$ **M1**

Therefore $\frac{8^9}{9^8} \approx 3^{1.01} \approx 3$ **A1 AG**

M1	Attempt to find a probability of a sequence of heads followed by a tail
A1	Correct expected value
M1	Consideration of how expected value changes with h
A1	Correct expression
M1	Justification that the expected value increases with h while $h < N$
M1	Justification that the expected value decreases with h while $h > N$
A1	Conclusion that winnings can be maximised if $h = N$
	Subtotal: 7
B1	Identifies a strategy for considering all winning sequences
M1	Correct probability for one case, could be seen as part of full sum
M1	Generalised to any case
M1	Expression as a sum and restatement so that binomial can be identified
A1	Fully correct expression
M1	Identification of binomial expansion
A1	Correct simplification
A1	Fully justified expression for expected winnings
	Subtotal: 8
B1	Identification of maximum value for expected winnings
M1	Takes logs and simplifies
M1	Further simplification
M1	Applies given approximation
A1	Concludes given estimate for expected winnings
	Subtotal: 5

Question 13

(i)	$A_1 = \frac{1}{2}, C_1 = 0$	**B1**
	$B_1 = \frac{1}{4}, D_1 = \frac{1}{4}$	**B1**
	$A_2 = \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{3}{8}$	**M1** **M1** **A1**
	$B_2 = D_2 = \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} = \frac{1}{4}$	**M1** **A1**
	$C_2 = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{1}{8}$	**M1** **A1**
(ii)	$B_{n+1} = \frac{1}{2}B_n + \frac{1}{4}(A_n + C_n)$	**M1**
	$A_n + B_n + C_n + D_n = 1$	**M1**
	$B_n = D_n$ (by symmetry)	**M1**
	Therefore $A_n + C_n = 1 - 2B_n$	**M1**
	$B_{n+1} = \frac{1}{4}$ and so $B_n = D_n = \frac{1}{4}$ for all n.	**A1**
	$A_{n+1} = \frac{1}{2}A_n + \frac{1}{4}(B_n + D_n) = \frac{1}{2}A_n + \frac{1}{8}$	**M1** **A1**
	$A_{n+1} - \dfrac{1}{4} = \dfrac{1}{2}\left(A_n - \dfrac{1}{4}\right)$	**M1**
	Therefore $\left(A_n - \frac{1}{4}\right)$ is a geometric sequence with common ratio $\frac{1}{2}$	**M1**
	$A_n = \frac{1}{4} + \left(\frac{1}{2}\right)^{n+1}$	**A1**
	$C_n = \frac{1}{4} - \left(\frac{1}{2}\right)^{n+1}$	**A1**

B1	Both values correct
B1	Both values correct
M1	One of the three cases identified in calculation or a tree diagram drawn to show all cases
M1	All three cases correctly identified
A1	Correct value
M1	Correct calculation
A1	Correct value
M1	Correct calculation
A1	Correct value
	Subtotal: 9
M1	Recurrence relation for B_n (or D_n) found
M1	Statement that probabilities add up to 1
M1	Identification of symmetry in problem or a recurrence relation to identify this relationship
M1	Combination so that A_n and C_n can be eliminated
A1	Correct value
M1	Recurrence relation for A_n
A1	Correct relation having substituted for B_n and D_n
M1	Appropriate method to find A_n
M1	Identification of geometric sequence
A1	Correct expression for A_n (must be simplified)
A1	Correct expression for C_n (must be simplified)
	Subtotal: 11

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